THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050 Mathematical Analysis (Spring 2018)
Tutorial on Apr 18

If you find any mistakes or typos, please email them to ypyang@math.cuhk.edu.hk

Part I: Some comments.

- For a uniformly continuous function $f : A \to \mathbb{R}$, δ can be chosen to depend only on ε and NOT on the points in A.

- Continuity itself is a **pointwise (local)** property of a function f, that is, f is continuous or not at a particular point, and this can be determined by looking at only the values of $f(x)$ in an (arbitrarily small) neighborhood of that point. When we speak of f being continuous on an interval, we mean only that f is continuous at every point of this interval.

  In contrast, uniform continuity is a **global** property in the sense that the definition refers to **pairs** of points rather than individual points. So we cannot say that whether f is uniformly continuous at some point $x \in A$.

  The mathematical statements that f is continuous on A and the definition that f is uniformly continuous on A are very similar. Please distinguish the following quantifications:

$$\begin{aligned} \text{continuous}: &\quad \forall x \in A\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall y \in A;\ |x-y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon, \\ \text{uniformly continuous}: &\quad \forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in A;\ |x-y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon. \end{aligned}$$

- **Nonuniform continuity criteria 5.4.2 (iii)** is very useful for proving that f is not uniformly continuous on A. Also refer to Question 3 below.

- **(Cantor's Theorem)** The **Uniform continuity Theorem 5.4.3** guarantees that a continuous function $f(x)$ on a closed bounded interval is uniformly continuous. However, when the interval is not closed and bounded, a continuous function can still be uniformly continuous. In particular, if f is defined on a bounded open interval (a, b), a condition for f to be uniformly continuous is given in **Theorem 5.4.8**: $\lim_{x \to a+} f(x)$, $\lim_{x \to b-} f(x)$ both exist and are finite.

- We have the following chain of inclusions for functions over a **closed bounded** subset of $\mathbb{R}$:

$$\text{Lipschitz continuous} \subset \text{uniformly continuous} = \text{continuous}$$

  Uniform continuity does not imply Lipschitz continuity. Please refer to **Ex 5.4.11** for a counterexample.

Part II: Exercises from the textbook.

1. **(Ex 5.4.7)** If $f(x) := x$ and $g(x) = \sin x$, show that both f and g are uniformly continuous on $\mathbb{R}$, but that their product fg is not uniformly continuous on $\mathbb{R}$.

   Remark: The statement will be true if f, g are defined on a bounded subset of $\mathbb{R}$.

Proof: Notice that

$$|\sin x - \sin y| = \left|2\cos\frac{x+y}{2}\sin\frac{x-y}{2}\right| \le \left|2\sin\frac{x-y}{2}\right| \le 2\left|\frac{x-y}{2}\right| = |x-y|$$

and thus f, g are both Lipschitz functions on $\mathbb{R}$ and consequently uniformly continuous.

Consider $x_n = 2n\pi + \frac{1}{n}$, $y_n = 2n\pi$, then $\lim_{n\to\infty}(x_n - y_n) = 0$ while

$$|(fg)(x_n) - (fg)(y_n)| = \left|\left(2n\pi + \frac{1}{n}\right)\sin\left(2n\pi + \frac{1}{n}\right)\right| = \left(2n\pi + \frac{1}{n}\right)\sin\frac{1}{n} \to 2\pi.$$

Therefore, fg is not uniformly continuous on $\mathbb{R}$.

2. In (b)-(d), determine whether the statement is true or false. If true, prove it; if false, give a counterexample.

   (a) **(Ex 5.4.10)** Prove that if f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A.

   (b) If f is continuous and bounded on a bounded subset A of $\mathbb{R}$, then f is uniformly continuous on A.

   (c) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$, then f is bounded on $\mathbb{R}$.

   (d) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded on $\mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.

   Remark: Notice that we do not require A to be a closed interval in (a). Also we cannot obtain boundedness if f is only continuous.

Part III: Additional exercises.

3. **(Question 10 on Mar 28 revisited)** Suppose A is a bounded subset of $\mathbb{R}$. Show that f is uniformly continuous on A if and only if for any Cauchy sequences in A, $(f(x_n))$ is also a Cauchy sequence.

   Proof: ($\Longrightarrow$) $\forall\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in A$, $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$. Suppose (x_n) is a Cauchy sequence in A, then $\exists N \in \mathbb{N}$ such that if $m, n \ge N$ then $|x_m - x_n| < \delta$ and consequently $|f(x_m) - f(x_n)| < \varepsilon$. Therefore, $(f(x_n))$ is a Cauchy sequence.

   ($\Longleftarrow$) Suppose f is not uniformly continuous, then $\exists\varepsilon_0 > 0$ and two sequences in A such that $\lim_{n\to\infty}|x_n - y_n| = 0$ while $|f(x_n) - f(y_n)| \ge \varepsilon_0$ for all n.

   Since A is bounded, so are (x_n) and (y_n). Then by Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Suppose $\lim_{k\to\infty} x_{n_k} = c$ (which does not need to be in A), then (y_{n_k}) also converges to c (think about why).

   Now we define a sequence $(z_k) = (x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, \cdots)$. It can be seen that (z_k) converges (to c) and thus is a Cauchy sequence. However, $(f(z_k))$ is not a Cauchy sequence because $|f(x_{n_k}) - f(y_{n_k})| \ge \varepsilon_0, \quad \forall k \in \mathbb{N}$.

   Therefore, f must be uniformly continuous on A.

Remarks: 1. The conclusion does not hold any more if A is unbounded. You can consider $f(x) = x^2$ on $A = \mathbb{R}$ as a counterexample.

2. We can also see that f is uniformly continuous if and only if for any sequences $(x_n), (y_n) \subset A$ (bounded or unbounded) with $\lim\limits_{n\to\infty}(x_n - y_n) = 0$, it holds that $\lim\limits_{n\to\infty}[f(x_n) - f(y_n)] = 0$. This necessary and sufficient condition is particularly useful for proving that some given function is not uniformly continuous. See Q1 and Q2d.

4. **(Generalization of Continuous Extension Theorem 5.4.8)** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lim\limits_{x\to-\infty} f(x) = L_1$, $\lim\limits_{x\to\infty} f(x) = L_2$ exist in $\mathbb{R}$.

   (a) Show that f is uniformly continuous on $\mathbb{R}$.

   (b) Is the converse of (a) true or false: f is uniformly continuous on $\mathbb{R} \implies$ both limits at infinity exist? (Compare with Theorem 5.4.8)

5. (b) and (c) are supplementary properties of periodic functions.

   (a) **(Ex 5.4.14)** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **periodic** on $\mathbb{R}$ if their exists a number $p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

   (b) p is called a period of f. If there exists a least positive constant T among the periods of $f(x)$, it is called the **fundamental (primitive, basic, prime) period**.

   - A continuous function may not have a fundamental period (constant function).
   - A non-constant function may not have a fundamental period. **Dirichlet function** is an example, for which any positive rational number is a period.
   - However, a non-constant continuous function must have a fundamental period.

   (c) The sum of two period functions may not be a periodic function. Consider $\sin x + \sin \pi x$.

Proof: **(a)** Notice that f is continuous and consequently uniformly continuous on $[0, p] \implies |f(x)| < M, \forall x \in [0, p]$ for some $M > 0$. So $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $x - np \in [0, p)$ and $|f(x)| = |f(x - np)| < M$.

Therefore, f is bounded on $\mathbb{R}$.

Also, $\forall \varepsilon > 0, \exists \delta_1 > 0$ such that if $x, y \in [0, p], |x - y| < \delta_1$ then $|f(x) - f(y)| < \dfrac{\varepsilon}{2}$. Let $\delta = \min(\delta_1, p)$. If $|x - y| < \delta$ (WLOG, we assume $x \leq y$), then there are two cases:

1°. $x, y \in [np, np + p]$ for some n, then $|f(x) - f(y)| = |f(x - np) - f(y - np)| < \dfrac{\varepsilon}{2}$.

2°. $x \in [np - p, np), y \in [np, np + p]$ for some n. Then

$$
\begin{aligned}
&= |f(x - np) - f(0)| + |f(y - np) - f(0)| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

In either case we have $|f(x) - f(y)| < \varepsilon$ and consequently f is uniformly continuous on $\mathbb{R}$.

6. **(Optional, compare with 4(b))** Suppose $f : A = [0, \infty) \to \mathbb{R}$ is uniformly continuous on A and $\lim\limits_{n\to\infty} f(n + h) = L$ for any $h \in [0, 1]$. Show that $\lim\limits_{x\to\infty} f(x) = L$.

Proof: $\forall \varepsilon > 0, \exists \delta > 0$ such that whenever $|x-y| < \delta$ it follows $|f(x) - f(y)| < \frac{\varepsilon}{2}$.

For this $\delta > 0$, we can take $m \in \mathbb{N}$ such that $m > \frac{1}{\delta} \Longrightarrow 0 < \frac{1}{m} < \delta$. Let $x_k = \frac{k}{m}$, $k = 0, 1, 2, \cdots, m$ and then from the assumption we have $\lim\limits_{n\to\infty} f(n + x_k) = L$, i.e., there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $|f(n + x_k) - L| < \frac{\varepsilon}{2}$.

Now $\forall x > N$ we can write $x = [x] + (x - [x])$. Since $x - [x] \in [0, 1)$, there exists k such that $|x - [x] - x_k| \leq \frac{1}{m} < \delta$. Therefore (notice that $[x] \geq N$),

$$|f(x) - L| \leq |f(x) - f([x] + x_k)| + |f([x] + x_k) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and we conclude that $\lim\limits_{x\to\infty} f(x) = L$.

7. Suppose $f(x)$ is a **Lipschitz continuous** on $[a, \infty)$, $a > 0$. Show that $\frac{f(x)}{x}$ is uniformly continuous on $[a, \infty)$.

Proof: **1**°. From assumption, there exists $M_1 > 0$ such that

$$|f(x) - f(y)| \leq M_1|x - y|, \forall x, y \geq a.$$

In particular,

$$\begin{aligned} |f(x) - f(a)| \leq M_1|x - a| &\Longrightarrow |f(x)| \leq M_1(x - a) + |f(a)| \\ &\Longrightarrow \frac{|f(x)|}{x} \leq M_1\frac{x - a}{x} + \frac{|f(a)|}{x} \leq M_1 + \frac{|f(a)|}{a}. \end{aligned}$$

2°. Therefore, for any $x, y \geq a$ we have

$$\begin{aligned} \left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| &= \left|\frac{yf(x) - xf(y)}{xy}\right| = \left|\frac{(y - x)f(x) - x(f(y) - f(x))}{xy}\right| \\ &\leq \frac{|y - x||f(x)| + x|f(y) - f(x)|}{xy} \\ &= \frac{|y - x|}{y} \cdot \frac{|f(x)|}{x} + \frac{|f(y) - f(x)|}{y} \\ &\leq \frac{|y - x|}{a} \cdot \frac{|f(x)|}{x} + \frac{|f(y) - f(x)|}{a} \\ &\leq \frac{|y - x|}{a} \cdot \left(M_1 + \frac{|f(a)|}{a}\right) + \frac{M_1|x - y|}{a} \\ &= M|x - y| \end{aligned}$$

where $M = \frac{2aM_1 + |f(a)|}{a^2}$. Therefore, $\frac{f(x)}{x}$ is Lipschitz continuous and consequently uniformly continuous on $[a, \infty)$.